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GENERALIZATION OF THE STATISTICAL THEORY OF STRENGTH FOR THE CA--ETC(U)

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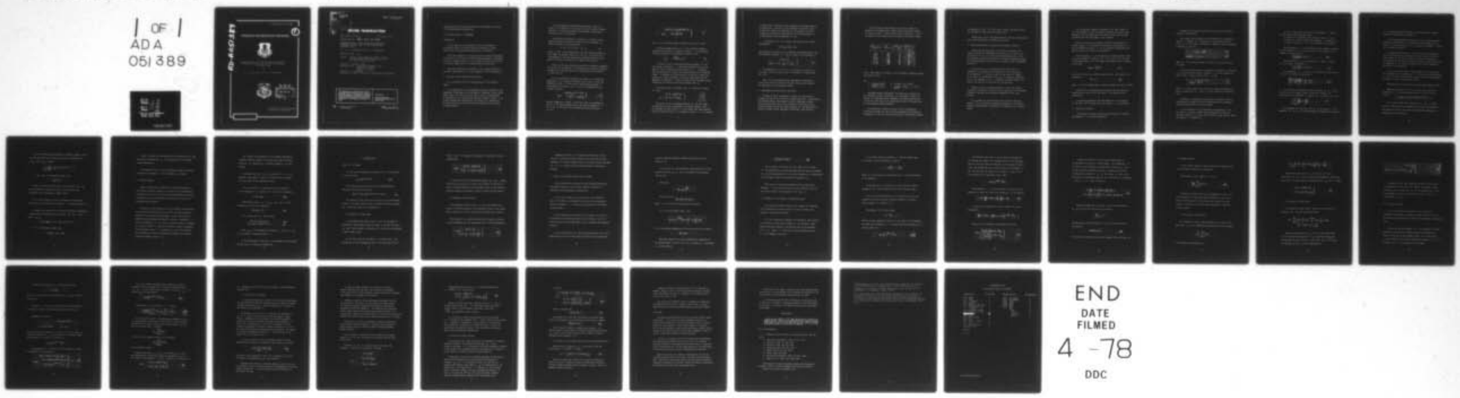

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of

T. A. Kontorova and O. A. Timoshenko

FOREIGN TECHNOLOGY DIVISION

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Generalization of the Statistical Theory of Strength for the Case of the Nonuniformly Stressed State

T.A. Kontorova and O.A. Timoshenko

Introduction

At one time one of the authors [1,2,3,4,5] developed a statistical theory of technical cohesive strength for solids for the case of a uniformly stressed state of a material.

Since the influence of the scale factor is observed experimentally not only in the case of extension and compression of specimens, but also in bending tests, it is of interest to extend this theory to the more common case of the nonuniformly stressed state.

This paper represents an attempt at such a generalization of the theory undertaken by us on the suggestion of N.N. Davidenkova.

1. The Case of the Uniformly Stressed State

Let us briefly recall the main postulates of the previous theory.

It is assumed that the experimentally observed influence of the dimensions of specimens on the strength of a material (the so-called "scale effect") is due to the fact that in each of the specimens is contained a set of defects with different "hazard" levels. It is further assumed that responsibility for failure of a specimen at any time is borne by only one--the most hazardous--defect present in this specimen.

As the parameter characterizing the "hazard" level of a defect it is convenient to select the value of technical cohesive strength F, which the specimen would possess if this defect were the source of its failure.

The possible "assortment" of defects in a material can be characterized by series of successive values of parameter F, which we arrange in increasing order:

$$F_1, F_2 \ldots F_{s-1}, F_s, F_{s+1} \ldots F_0 \ldots F_l, \tag{1}$$

where F_1 and F_ℓ are respectively the lower and upper limits of the "range" of possible values of F, and F_0 is the value of F corresponding to the most frequently encountered defects in the material in question (defects of a "middle" hazard level).

Since only the most hazardous defects are considered responsible for failure of the specimens, in determining the technical cohesive strength of the material we will be interested only in values of F lower than F_0.

If it is assumed that deviation of values of F around the magnitude of F_0 follows the ordinary law of "errors," then the probability, $p(F_i)$, of the defect's encountering parameter F lying within the range of F_i to $F_i + dF$ is determined by the Gaussian function:

$$\left.\begin{array}{l} p(F_i)\,dF = Ce^{-\alpha(F_i - F_0)^2}\,dF \\ \text{where} \quad C = \sqrt{\frac{\alpha}{\pi}}, \quad \alpha = \frac{1}{2(F-F_0)^2} \end{array}\right\}, \tag{2}$$

and the probability, $W(F)dF$, of the fact that in a specimen of volume V the most hazardous defect will have a parameter of F to $F + dF$ is written in the following form (more precisely, cf. [2]):

$$
\left.\begin{aligned}
&W(F)\,dF = \bar{n}V\cdot p(F)[I(F)]^{\bar{n}V-1}\,dF\\
&\text{where}\qquad I(F)=\int_F^{\infty} p(F)\,dF
\end{aligned}\right\}, \qquad (3)
$$

and n is the average number of defects per unit of volume.

The most probable value, F* , of the technical cohesive strength of specimens of a specific volume of V is determined from the condition of the maximum of function W(F):

$$
\left[\frac{\partial W(F)}{\partial F}\right]_{F=F^*}=0. \qquad (4)
$$

A distinct form of relationship between strength F* and the volume of specimens V , was found previously only for the specific case of sufficiently high values of V . In seeking this relationship the authors of [2] were guided by the following ideas. In specimens of great volume it is natural to assume the presence of a great assortment of defects; this means that the magnitude of parameter F characterizing the most hazardous defect responsible for failure must be much smaller than F_0 .

With high values of difference $(F_0 - F)$, equation (4) gives (cf. [2]):

$$
\left.\begin{aligned}
&F^* \cong F_0-\sqrt{A\lg V+B}\\
&\text{where}\quad A=\frac{1}{\alpha},\quad B=\frac{1}{\alpha}\lg\frac{\bar{n}}{2\sqrt{\pi}}
\end{aligned}\right\}. \qquad (5)
$$

The nature of the relationship between F* and the volume of specimens can also be determined for another specific case-- for specimens of small volume V , if we assume that small possible values of difference $(F_0 - F)$ correspond to small V . It is natural

to assume that a reduction in the dimensions of specimens must be accompanied by a reduction in the assortment of defects present in them; the smaller the dimensions of the specimen, the less frequently are defects encountered in it for which parameter F is much smaller than the magnitude of F_0[1].

If difference $(F_0 - F)$ is small, then interval I(F) equals approximately

$$I(F) \cong \frac{1}{2} + C(F_0 - F),$$

and equation (4) results in the following relationship between the most probable strength, F^*, and the volume of the specimens, V:

$$\left.\begin{aligned} F^* &\cong a + \frac{b}{V} \\ \text{where} \quad a &= F_0 - \frac{1}{2\sqrt{\alpha}}, \quad b = \frac{\sqrt{\pi}}{2\sqrt{\alpha}\,n} \end{aligned}\right\} \tag{6}$$

The dependence of F^* on V, as we shall see, proves in this case to be more critical than for specimens of large volume (cf. (5)).

This fact is in agreement with experimental observations--as we know, the scale factor is much more important the smaller the dimensions of the specimens studied.

2. Experimental Verification of the Theory

It would be quite interesting to verify the correctness of formulas (5) and (6) on the basis of experimental data. Unfortunately, the possibilities in this regard are still exceedingly limited. We found quantitative data on the influence of the scale factor during extension only in the study by Müller [6], who measured the tensile strength of NaCl specimens of different cross section.[2]

In the following table Müller's data (third column) are compared with strength values computed from theoretical formula (6) (fourth column). Points used to compute constants a and b in formula (6) are marked with an asterisk in the second column.

(1) Сечение образца, мм^2	(2) Объем V, см^3	(3) Прочность $F_{эксп.}$, кг/см^2	(4) Прочность $F_{теор.} = a + \frac{b}{v}$ кг/см^2
6 × 6	1.26*	40	40
4 × 4	0.56	42	44.3
3 × 3	0.315	47	50.5
2.5 × 2.5	0.22*	56	56
2 × 2	0.14	72	68
1.5 × 1.5	0.084	89	89
1 × 1	0.035	200	162.5
0.7 × 0.7	0.017	350	296.5
0.5 × 0.5	0.008	600	588

Note: *The length of specimens in all experiments remained constant and equaled 35 mm.

Key:

1. Cross section of specimen, mm^2
2. Volume V , cm^3
3. Strength F_{exp}, kg/cm^2
4. Strength $F_{theor.} = a + (b/v)$, kg/cm^2

Agreement between experimental and theoretical results can be considered totally satisfactory, in our opinion. Formula (5) results in considerably worse agreement with experimental values in this case, and this is not surprising in view of the small size of the specimens studied by Müller.

But formula (5), which is correct only for specimens of rather large volume, has also received experimental confirmation. In Nature [7] a curve is given, expressing the relationship between

the magnitude of $(F_0 - F)^2$ and log V, plotted from Brown's data for specimens of hard coal of different size.

Experimental points are positioned quite well on the theoretical line, which proves the correctness of rule (5).

3. Taking into Account the Spatial Distribution of Defects

The method developed above, as we will see, proves to be totally suitable for considering the question of the role of the scale factor in the case of the uniformly stressed state. But, in attempting to apply this method to solving the analogous problem for the nonuniformly stressed state we encounter quite considerable difficulty.

The key feature of the nonuniformly stressed state is unevenness of distribution of stresses over the cross section of the specimen. It is obvious that in this case the "hazard" of any defect is determined not only by its intrinsic "qualities" (e.g., its dimensions or the F parameter introduced above), but also by the position of this defect in the specimen.

Because of this it becomes necessary to take into account the spatial distribution of defects in the material of the specimen. A method of solving this problem was suggested by Prof. Ya.I. Frenkel'.

If we assume that the defects do not "interact" with one another, then the problem is completely analagous to the familiar problem of distribution of particles of an ideal gas in a certain volume.

Let us consider a group of specimens of the same volume V. If it is possible to count the number of defects in each of these specimens, then in going from one specimen to the other we would certainly find that this number is not constant.

Only on average can the number of defects in specimens of a certain volume V , be considered constant and proportional to V; the true number of defects N , in each specimen individually will differ from the mean value of this number $\overline{N}$.

The probability, P , of encountering a certain specific fluctuation in the number of defects, $\Delta N = N - \overline{N}$, just as in the case of the gas problem, can be determined approximately with a function of the type:

$$P(N) \simeq \frac{e^{-\overline{N}}(\overline{N})^N}{N!}, \qquad (7)$$

$\overline{N}$ we will in this case consider proportional to the volume of the specimen, V,

$$\overline{N} = \bar{n} V, \qquad (8)$$

where $\bar{n}$ is the average number of defects present per unit of volume.

Function (7) in combination with equation (8) describes the distribution of defects over the volume of the material which interests us.

It should be emphasized that approximation (7) is suitable at any level, both for high and as low values of V as possible.

4. "Quality" of Defects

As previously, we will characterize the "quality" of defects by parameter F , as defined previously.

Equations (7) and (8) can be written for each sort of defect present in the material.

For example, the probability of encountering in a specimen of volume V a number, N_i, of defects of a certain specific i-th sort, i.e., defects which correspond to a specific value of parameter F equaling F_i, will be:

$$P_i(N_i) \cong \frac{e^{-\bar{N}_i}\bar{N}_i^{N_i}}{(N_i)!}, \tag{7'}$$

$$\text{whereby} \quad \bar{N}_i = \bar{n}_i \cdot V, \tag{8'}$$

where $\bar{n}_i$ is the average number of defects of this sort per unit of volume.

As far as the distribution of defects by "quality" is concerned, i.e., by the possible values of parameter F, we will assume, as before, that it obeys the Gauss law, i.e., that

$$\left.\begin{aligned} \bar{n}_i &= Ke^{-\alpha(F_i - F_0)^2} \\ \text{where} \quad K &= C\bar{n} \end{aligned}\right\}, \tag{9}$$

where $\bar{n}$ is the average value of the total number of defects per unit of volume, and C, α, and F_0 are as previously defined (cf. (2)).

As always, in considering the question of possible values of the material's strength we will be interested only in parameters characterizing defects of the worst grade.

In this connection let us determine the probability of the fact that in a specific volume under consideration a defect of the worst "quality" is a defect characterized by some specific value of parameter F equaling F_s.

If F_s is the lowest of all values of parameter F present in this volume, then this means that:

1) There are no defects in this volume for which $F < F_s$.

2) There is a defect present with parameter F equaling F_s.

3) Defects for which $F > F_s$ can be present in any amount.

The probability, ϕ_i, of the absence in this volume of defects of some specific, e.g., i-th sort we will find by making N_i in formula (7') equal to zero:

$$\varphi_i = P_i(N_i = 0) = e^{-\bar{N}_i}. \tag{10}$$

The probability, ψ_s, of the presence of a defect for which $F = F_s$ is equal, accordingly, to

$$\psi_s = P_s(N_s = 1) = \bar{N}_s \cdot e^{-\bar{N}_s}. \tag{11}$$

The probability, χ_i, of the fact that defects of some specific i-th sort are present in any amount is

$$\chi_i = \sum_{N_i=0}^{\infty} P_i(N_i) = \sum_{N_i=0}^{\infty} \frac{e^{-\bar{N}_i}\bar{N}_i^{N_i}}{(N_i)!} = 1. \tag{12}$$

And, finally, the probability which interests us, W_s, of the fact that in this volume a defect of the worst "quality" is characterized by a parameter F equal to F_s is equal to (cf. (1)):

$$W = \prod_{i=1}^{s-1} \varphi_i \cdot \psi_s \cdot \prod_{i=s+1}^{l} \chi_i. \tag{13}$$

The dependence of factors ϕ_i and ψ_s on the volume V, and parameters F_i and F_s is then determined by equations (8') and (9).

5. Correlation Between Dimensions of the Specimen and Possible Amounts of Fluctuation in Parameter F

It has been mentioned more than once that in considering the question of the strength of a material we are interested only in defects of the worst quality, i.e., in values of parameter F which are the smallest possible.

We assumed above in para. 1 that in the case of specimens of rather large volume the assortment of defects present in them is quite varied, as the result of which the values of parameter F corresponding to defects of the worst quality are much smaller than the magnitude of F_0 characterizing the defects most frequently encountered in the material.

In the case of specimens of small volume, on the other hand, we considered that fluctuations of F from its most probable value, F_0, can not take on large values, since in this case the assortment of defects can not be sufficiently varied.

The qualitative ideas can now be substantiated quantitatively.

Employing equations (7'), (8'), and (9), let us write the probability, ψ_i, of encountering a defect of the i-th sort in volume V:

$$\psi_i = P_i(N_i = 1) = KVe^{-\alpha(F_i - F_0)^2} e^{-kVe^{-\alpha(F_i - F_0)}}$$

Let us first assume that fluctuation $|F_i - F_0|$ is great, and let us explain how the probability will vary of encountering some specific value of this great fluctuation when changing the dimensions of the specimens.

If we are dealing with specimens of different volume V_1 and V_2, then the ratio of the respective values of probabilities $(\psi_i)_{V_1}$ and $(\psi_i)_{V_2}$ equals:

$$\frac{(\psi_i)_{V_1}}{(\psi_i)_{V_2}} = \frac{V_1}{V_2} e^{-K(V_1 - V_2)e^{-\alpha(F_i - F_0)^2}}.$$

If $|F_i - F_0|$ is sufficiently great, then

$$\frac{(\psi_i)_{V_1}}{(\psi_i)_{V_2}} \cong \frac{V_1}{V_2}.$$

Hence it follows that high values of fluctuation $|F_i - F_0|$ are encountered in specimens of great volume actually more frequently than in specimens of small volume.

Let us now consider the case of specimens of small volume. What values can fluctuations of parameter F take on with small V ?

To answer this question let us study the behavior of probability ψ_i when varying the magnitude of difference $(F_i - F_0)$. Let us write log ψ_i:

$$\lg \psi_i = \lg(KV) - \alpha(F_i - F_0)^2 - KVe^{-\alpha(F_i - F_0)^2}.$$

If V is sufficiently small, then

$$\lg \psi_i \cong -\alpha(F_i - F_0)^2.$$

Hence it follows that the greater the fluctuation in $(F_i - F_0)$, the less the probability, ψ_i , of encountering it in the small volume considered, V.

This means that in the case of specimens of small volume the fluctuation in parameter F can be assumed to be slight.

6. Tensile Strength

Before considering the question of the possible values of the technical cohesive strength of a material in the nonuniformly stressed state, we will use the notions described above to analyze the simpler case, and one studied by us earlier, of failure of specimens under extension.

As before, we will assume that in the uniformly stressed state of the material the hazard level of each of the defects does not depend on its position in the specimen. In this case the only magnitude characterizing the "hazard" of defects is parameter F , and the probability, W_s (cf. (13)), of the fact that in a specimen of a certain volume V , a defect of the worst quality corresponds to some specific value of F equaling F_s , is at the same time the probability that when this specimen is extended its technical cohesive strength is equal to F_s .

The problem of determining the most probable magnitude of technical cohesive strength as a result boils down to finding a value of F_s equaling $F_s{}^*$, whereby probability W_s has a maximum.

As follows from (13), W_s is a function of s, i.e., a function of the number of the sort of defect which is assumed to be the source of the specimen's failure.

It is not difficult to demonstrate that the condition for the maximum of W_s equaling $f(s)$ results in the equation

$$\varphi_s \psi_{s+1} = \psi_s \chi_{s+1}. \tag{14}$$

Substituting values ϕ_s, ψ_s, ψ_{s+1}, and χ_{s+1} in this equation, we arrive at the equation:

$$\overline{N}_s' = \overline{N}_{s+1}\, e^{-\overline{N}_{s+1}}. \tag{15}$$

Let us recall that (cf. (8') and (9)):

$$\overline{N} = \bar{n}_s V = KVe^{-\alpha(F_s - F_0)^2}, \tag{16}$$

$$\overline{N}_{s+1} = \bar{n}_{s+1} V = KVe^{-\alpha(F_{s+1} - F_0)^2}. \tag{16'}$$

Here F_{s+1} is the magnitude of parameter F next to $F = F_s$ in the "range" of possible values of F.

If the "assortment" of defects in our specimens is sufficiently varied, then it is natural to assume that

$$F_{s+1} \simeq F_s + \Delta F,$$

where ΔF is slight.

In this case, disregarding the square of ΔF in the exponent in (16'), we have

$$\bar{N}_{s+1} \simeq \bar{N}_s e^{-2\alpha (F_s - F_0) \cdot \Delta F}. \qquad (16'')$$

After substituting (16) and (16") and logarithmization, equation (15) acquires the form of

$$2\alpha (F_s^* - F_0) \cdot \Delta F + KVe^{-\alpha (F_s^* - F_0)^2 - 2\alpha (F_s^* - F_0)\Delta F} = 0. \qquad (17)$$

The solution to this equation should give us the relationship sought, of the most probable value of technical cohesive strength, F_s^*, versus the volume of the specimens V.

a) Specimens of Large Volume

It was demonstrated above in para. 5 that for specimens of sufficiently large volume we have the right to consider difference $|F_0 - F_s^*|$ which appears in equation (17) great, and for specimens of small volume, small.

Let us first study the case when V is rather great. Logarithmizing (17) and disregarding terms $\log [2\alpha(F_0 - F_s^*) \Delta F]$ and

$2\alpha(F_s^* - F_0)\ \Delta F$ as compared with magnitude $\alpha(F_s^*-F_0)^2$, we have approximately:

$$\left.\begin{aligned} &F_s^* \cong F_0 - \sqrt{A \lg V + B} \\ \text{where}\quad &A = \frac{1}{\alpha},\quad B = \frac{1}{\alpha}\lg K = \frac{1}{\alpha}\lg\frac{\sqrt{\alpha}n}{\sqrt{\pi}} \end{aligned}\right\}. \tag{18}$$

As we shall see, this relationship between F_s^* and V agrees with accuracy as great as a constant with formula (5) found by us earlier, which determines the most probable values of the technical cohesive strength of specimens of great size when tensile testing.

b) Specimens of Small Volume

For specimens of small volume. i.e., with sufficiently low values of difference $(F_0-F_s^*)$, the exponential factor in the second term of equation (17) can be conveniently expanded into a series.

This brings us to a quadratic equation for $(F_0-F_s^*)$; solving it and disregarding the term containing $(\Delta F)^2$, we get approximately

$$\left.\begin{aligned} &F_s^* \cong a + \frac{b}{V} \\ \text{where}\quad &a = F_0 - \frac{1}{\sqrt{\alpha}},\quad b = \frac{\Delta F}{K} = \frac{\sqrt{\pi}\Delta F}{\sqrt{\alpha}n} \end{aligned}\right\}. \tag{19}$$

Comparing (19) and (6) we become convinced that in this case--with specimens of small volume--we get precisely the same dependence of the most probable value of technical cohesive strength, F_s^*, on V, as when using the previous method of solving the problem.

7. Case of a Sufficiently Small Unit of Volume

Let us now proceed to solving our main problem--determining the possible values of the technical cohesive strength of a material in the nonuniformly stressed state.

In this case, unlike in the previous one, the "hazard" of each of the defects is determined not only by the corresponding values of parameter F, but also by the position of this defect in the specimen.

We will assume that the volume of the specimen as a whole can be divided into sufficiently small units of volume, so that the stress acting in each of these units can be considered approximately constant.

In this case function W_s which we discussed above (cf. (13)), being written for one of these units, will give us the probability

that the technical cohesive strength of this unit will be equal to F_s .

Let us write W_s more precisely, substituting in (13) the values of factors ϕ_i, ψ_s, and χ_i, determined from equations (10) to (12).

This gives

$$W_s = \overline{N}_s e^{-\sum_{i=1}^{s} \overline{N}_i}$$

Here (cf. (8')):

$$\overline{N}_i = n_i \delta V, \quad \overline{N}_s = n_s \delta V,$$

where δV is the unit of volume which we selected.

If δV is sufficiently small, then

$$W_s = n_s e^{-\delta V \sum_{i=1}^{s} \overline{n}_i} \delta V \cong n_s \left(1 - \delta V \sum_{i=1}^{s} n_i \right) \delta V,$$

or, if we disregard magnitudes of the second order of triviality:

$$W_s \cong n_s \delta V.$$

Employing equation (9), which determines the dependence of the average number of defects in a unit of volume, $\overline{n}_s$, on parameter F_s , we have finally

$$W_s \cong K e^{-a(F_s - F_0)^2} \delta V. \tag{20}$$

As is obvious, in the case of a very small unit of volume, δV , the probability of encountering some specific value of strength, F_s , boils down essentially to the apriori probability of encountering in this volume a defect characterized by parameter F_s .

This result it totally intelligible from the qualitative viewpoint. It can also be arrived at easily within the scope of our previous theory, from equation (3) (cf. para. 1).

8. Formulation of the Problem of Bending Strength

As a specific problem relating to the strength of a material in the nonuniformly stressed state let us first consider the problem of failure while bending.

If it is a question of specimens of rectangular cross section it is natural to select unit volumes, δV , in the form of thin layers positioned parallel to the neutral line of the specimen: $\delta V = S \cdot dz$, where S is the area of the specimen's base and dz is the height of the layer.

In the layer located at distance z from the neutral line, as we know, a stress acts which is equal to

$$\sigma = \frac{\sigma_0 z}{h},$$

where σ_0 is the stress at the surface and h is the half height of the specimen.

If the value of σ is equal here to the technical cohesive strength of the layer in which it acts, then the layer fails.

Let us recall that function W_s in the form of (20) is the probability that the technical cohesive strength of a certain unit of volume δV is equal to F_s.

Consequently, if in (20) we make

$$F_s = \frac{\sigma_0 z_s}{h},$$

then we get the probability that one of the layers of our specimen-- the layer with coordinate z_s-- will fail under the influence of the stress acting in it:

$$W_s = Ke^{-\alpha\left(\frac{\sigma_0 z_s}{h} - F_0\right)^2} \delta V. \tag{20'}$$

For precisely this layer to be the source of failure for the specimen as a whole it is necessary that all the remaining layers do not fail under the influence of the stresses acting in them. The probability that a certain layer with coordinate z_i will not fail with the presence of a stress of $\sigma_0 z_i/h$ in it is determined by the expression $(1-W_i)$, where

$$W_i = K e^{-\alpha\left(\frac{\sigma_0 z_i}{h} - F_0\right)^2} \delta V.$$

The probability, P_s , that the source of failure for the specimen as a whole is a layer with coordinate z_s will then be

$$P_s = W_s \cdot \prod_{i \neq s} (1 - W_i) = \frac{W_s}{1 - W_s} \prod_i (1 - W_i). \qquad (21)$$

In view of the triviality of unit of volume δV, we set, approximately

$$\prod_i (1 - W_i) \cong 1 - \sum_i W_i = 1 - KS \int_0^h e^{-\alpha\left(\frac{\sigma_0 z}{h} - F_0\right)^2} dz.$$

Function (21) is rewritten accordingly in the form

$$\left.\begin{aligned} P_s &= [1 - KSI(\sigma_0, h)] \frac{W_s}{1 - W_s} \\ \text{where} \quad I(\sigma_0, h) &= \int_0^h e^{-\alpha\left(\frac{\sigma_0 z}{h} - F_0\right)^2} dz \end{aligned}\right\} \qquad (21')$$

It makes no difference to us in exactly which layer of our specimen the process of failure begins. The probability, P, of failure of the specimen as a whole with a stress of σ_0 on its surface (without indicating the focus of failure) we will find by adding the probability, P_s, for all layers, or, what is the same thing, by integrating equation (21') with respect to coordinate z from 0 to h. This will give us

$$\begin{aligned} P = \sum_s P_s = [1 - KSI(\sigma_0, h)] \sum_s \frac{W_s}{1 - W_s} \cong \\ \cong [1 - KSI(\sigma_0, h)] \sum_s W_s = [1 - KSI(\sigma_0, h)] KSI(\sigma_0, h). \end{aligned} \tag{22}$$

The most probable value of stress σ_0 (we will designate it by σ_0^*) we will find from the usual condition

$$\left(\frac{\partial P}{\partial \sigma_0}\right)_{\sigma_0 = \sigma_0^*} = 0.$$

Differentiating (22) with respect to σ_0, we arrive finally at the equation

$$2KSI(\sigma_0^*, h) = 1, \tag{23}$$

the solution to which will give us the unknown value of stress σ_0^*.

9. Bending Strength

Let us convert integral $I(\sigma_0^*, h)$ which enters equation (23) to a form more convenient for integration.

Substituting $\sqrt{\alpha}(F_0 - \sigma_0^* z/h) = x$, we get

$$\frac{2KSh}{\sqrt{\alpha}\sigma_0^*} \int_{\sqrt{\alpha}(F_0-\sigma_0^*)}^{\sqrt{\alpha}F_0} e^{-x^2} dx = 1. \qquad (23')$$

Generally an explicit form of the dependence of stress σ_0* on the dimensions of specimens, which interests us, can not be found from this equation. As before we are limited, for this reason, to studying two specific cases--specimens of large and of small volume.

a) Specimens of Large Volume

For specimens of large volume difference $(F_0 - \sigma_0^*)$, as we know (para. 5), can be considered sufficiently great, and integral

$$I_0 = \int_{\sqrt{\alpha}(F_0-\sigma_0^*)}^{\sqrt{\alpha}F_0} e^{-x^2} dx$$

can be computed approximately as

$$I_0 = \int\limits_{\sqrt{\alpha}(F_0-\sigma_0^*)}^{\infty} e^{-x^2}\,dx - \int\limits_{\sqrt{\alpha}F_0}^{\infty} e^{-x^2}\,dx \cong \frac{e^{-\alpha(F_0-\sigma_0^*)^2}}{2\sqrt{\alpha}(F_0-\sigma_0^*)} - \frac{e^{-\alpha F_0^2}}{2\sqrt{\alpha}F_0}.$$

Substituting this value of I_0 in equation (23') and substituting $Sh = V/2$, after uncomplicated mathematical transforms we arrive at the approximate relationship between σ_0^* and V sought:

$$\left.\begin{aligned} \sigma_0^* &\cong F_0 - \sqrt{A \lg V + B_b} \\ A &= \frac{1}{\alpha}, \quad B_b = \frac{1}{\alpha} \lg \frac{\pi}{2\sqrt{\pi\alpha}F_0} \end{aligned}\right\}. \tag{24}$$

b) Specimens of Small Volume

For specimens of small volume, employing the triviality of difference $(F_0 - \sigma_0^*)$, we have approximately

$$I_0 = \int\limits_{\sqrt{\alpha}(F_0-\sigma_0^*)}^{\sqrt{\alpha}F_0} e^{-x^2}dx = \int\limits_0^{\infty} e^{-x^2}\,dx - \int\limits_0^{\sqrt{\alpha}(F_0-\sigma_0)} e^{-x^2}dx - \int\limits_{\sqrt{\alpha}F_0}^{\infty} e^{-x^2}\,dx \cong$$
$$\cong \frac{\sqrt{\pi}}{2} - \sqrt{\alpha}\,(F_0 - \sigma_0^*) - \frac{e^{-\alpha F_0^2}}{2\sqrt{\alpha}F_0}.$$

Substituting this value of I_0 in (23') and substituting $Sh = V/2$, solving the equation for σ_0^* obtained thereby, and disregarding the small values of $(1/2)\sqrt{\pi/\alpha}$ and $e^{-\alpha F_0^2}/2\sqrt{\alpha}F_0$ in comparison with F_0 , we find approximately

$$\left.\begin{aligned} \sigma_0^* &\simeq \frac{F_0}{1-\frac{1}{KV}} \\ \text{where (cf. (9) and (2))} \quad K &= \frac{\bar{n}\sqrt{\alpha}}{\sqrt{\pi}} \end{aligned}\right\} . \qquad (25)$$

Furthermore, $1/KV$ will always be a proper fraction, since for specimens of finite size $KV = \sqrt{\alpha}\bar{n}V/\sqrt{\pi}$ is certainly greater than unity (let us recall that $\bar{n}V$ is the average value of the total number of defects in a specimen of volume V, and $\alpha = (1/2)\overline{(\Delta F)^2}$ (cf. (2)).

10. Torsional Strength

Let us now consider the question of the influence of dimensions of specimens on strength in another specific case of the nonuniformly stressed state--twisting of specimens of cylindrical form.

In this case as unit volumes, δV, it is convenient to choose sufficiently thin hollow cylinders positioned coaxially with respect to the axis of the specimen, where $\delta V = 2\pi Lrdr$, where r is the distance of the unit cylinder from the axis, dr is its thickness, and L is the height of the specimen.

In each of these units, δV , a shearing stress acts:

$$\tau = \tau_0 \frac{r}{R};$$

Here R is the radius of the specimen and τ_0 is the stress on its surface.

As soon as τ is equal to the technical cohesive strength of the unit of volume under consideration the latter fails.

Consequently, function

$$W_s = Ke^{-\alpha(F_s - F_0)^2} \delta V$$

(cf. (20)) where $F_s = \tau_0 \frac{r_s}{R}$

will represent the probability of failure of a unit of volume δV found at a distance of r_s from the axis of the cylinder, under the influence of a stress of τ acting in it:

$$W_s = Ke^{-\alpha\left(\tau_0 \frac{r_s}{R} - F_0\right)^2} \delta V.$$

The probability, P , of failure of the specimen as a whole is determined by the expression

$$\left.\begin{aligned} P &\cong [1 - K2\pi LI(\tau_0, R)] \sum_s \frac{W_s}{1 - W_s} \cong \\ &\cong [1 - K2\pi LI(\tau_0, R)] K2\pi LI(\tau_0, R) \\ \text{where} \quad I(\tau_0, R) &= \int_0^R e^{-\alpha\left(\tau_0 \frac{r}{R} - F_0\right)^2} r dr \end{aligned}\right\} \qquad (26)$$

As in the problem considered above, relating to bending strength, the most probable value of the shear breaking point, $\tau_0 = \tau_0^*$, can be found from the condition $(\partial P/\partial \tau_0)_{\tau_0 = \tau_{0*}} = 0$, which results in the equation

$$4\pi LK\int_0^R e^{-\alpha\left(\tau_0^*\frac{r}{R}-F_0\right)^2} r\,dr = 1. \qquad (27)$$

Assuming that $\sqrt{\alpha}[\tau_0^*(r/R)-F_0] = x$, we have

$$\frac{4\pi LKR^2}{\sqrt{\alpha}(\tau_0^*)^2}\left[F_0\int_{\sqrt{\alpha}(F_0-\tau_0^*)}^{\sqrt{\alpha}F_0} e^{-x^2}dx+\frac{1}{\sqrt{\alpha}}\int_{\sqrt{\alpha}(F_0-\tau_0^*)}^{\sqrt{\alpha}F_0} e^{-x^2}x\,dx\right]=1. \qquad (27')$$

We are restricted here to considering the dependence of τ_0^* on the dimensions of specimens for the specific case of twisting specimens of sufficiently large volume, when difference $(F_0-\tau_0^*)$ can be considered great. Integral

$$I_1=\int_{\sqrt{\alpha}(F_0-\tau_0^*)}^{\sqrt{\alpha}F_0} e^{-x^2}dx$$

for this case was computed in para. 9; and integral

$$I_2=\int_{\sqrt{\alpha}(F_0-\tau_0^*)}^{\sqrt{\alpha}F_0} e^{-x^2}x\,dx$$

can be computed directly.

Substituting values I_1 and I_2 in equation (27'), after some uncomplicated computations we get the dependence of τ_0^* on the volume of the specimens, $V = \pi R^2 L$, which we are interested in:

$$\left.\begin{aligned}&\tau_0^*\cong F_0-\sqrt{A\lg V+B_{\pm}}\\ \text{where}\quad &A=\frac{1}{\alpha},\; B_{\pm}=\frac{1}{\alpha}\lg\frac{2n}{\sqrt{\pi\gamma}}\end{aligned}\right\}. \qquad (28)$$

11. Influence of the Scale Factor on Strength in the Nonuniformly Stressed State

a) Scale Factor in Bending

It follows from equations (24) and (25) that both for specimens of large and small volume the probable values of technical cohesive strength while bending, σ_0^* , must be reduced with an increase in the volume of the specimens, V .

Furthermore, as in the case of extension, the dependence of σ_0^* on V proves to be more critical for specimens of small volume. Comparing (24) with formulas (5) and (18) obtained by us earlier for technical cohesive strength during extension, we come to the conclusion that in the case of specimens of large volume the dependence of technical cohesive strength on V is of the same nature both when bending and during extension: The change in strength accompanying a change in V is proportional to the square root of the logarithm of V .

Let us write equation (25), determining technical cohesive strength while bending for specimens of small volume, in the form

$$\sigma^*_0 \cong \frac{F_0}{1-\frac{1}{KV}} \cong F_0\left(1+\frac{1}{KV}\right) \qquad (25')$$

(in view of the triviality of 1/KV , we are limited to the first two terms of the expansion of σ_0^* into a series).

Comparing (25') with (6), we become convinced that in the case of specimens of small volume the nature of the dependence of technical cohesive strength on V during extension and during bending is practically the same.

It would be highly desirable, of course, to be able to subject equations (24) and (25), which describe the influence of the scale factor on the technical cohesive strength of a material when bending, to experimental verification.

However, we have not found appropriate experimental data in the literature. We know of only one study [8] in which values of technical cohesive strength while bending were measured for specimens of different dimensions. The numerical data presented in this study, which concerns only three different values of V, unfortunately, however, is insufficient for verification of theoretical formula (24), which contains three unknown constants (F_0, A, and B_b).

We know that, all other experimental conditions being equal, the technical cohesive strength of a material while bending is usually greater than its technical cohesive strength measured during extension.

Let us attempt to compare from this viewpoint the possible values of σ_0* and F_0* determined from formulas (24) and (18), respectively.

Constants F_0 and A are identical in both cases, but constants B and B_b are different. Actually,

$$B=\frac{1}{\alpha}\lg\frac{\bar{n}\sqrt{\alpha}}{\sqrt{\pi}},$$

$$B_b=\frac{1}{\alpha}\lg\frac{\bar{n}}{2\sqrt{\pi\alpha F_0}},$$

i.e.,

$$B_b=B-\frac{1}{\alpha}\lg 2\alpha F_0.$$

Substituting this value of B_b in (24) and substituting $\alpha = 1/2(\overline{\Delta F})^2$ (cf. (2)), we have

$$\left.\begin{aligned} F_0^* &\simeq F_0 - \sqrt{A \lg V + B}, \\ \sigma_0^* &\simeq F_0 - \sqrt{A \lg V + B - 2\overline{(\Delta F)^2} \lg \frac{F_0}{\overline{(\Delta F)^2}}} \end{aligned}\right\} \quad (29)$$

Hence it follows that with a specific value of V, σ_0^* is actually greater than F_0^*. It is obvious further that ratio σ_0^*/F_0^* must not be much greater than unity, since term $2\,(\overline{\Delta F^2}) \log [F_0/(\overline{\Delta F^2})]$ can not be great.

It is natural to expect that the mean value of the square of the fluctuation in strength, $(\Delta F)^2$, will be less the more uniform the material. Consequently, ratio σ_0^*/F_0^* is a function of the "quality" of the material; it should be closer to unity, the more uniform the material studied. This conclusion, as we know, agrees with experimental observations.

b) Scale Factor When Twisting

From equation (28), which determines the dependence of technical cohesive strength of a material when twisting, τ_0^*, on the volume of specimens V, it follows that also when twisting an increase in the volume of the specimens should be accompanied by a reduction in the values of technical cohesive strength observed.

Furthermore, in the case of twisting specimens of sufficiently great volume considered by us the nature of the dependence of strength τ_0^* on V is precisely the same as in the case of extension and bending: The change in τ_0^* is proportional to the square root of the logarithm of V. Constant B_t which enters (28) is different, however, from the corresponding constants B and B_b in equations (18) and (24), which determine technical cohesive strength during extension (F_0^*) and bending (σ_0^*).

Actually,

$$B=\frac{1}{\alpha}\lg\frac{\bar{n}\sqrt{\alpha}}{\sqrt{\pi}},\quad B_t=\frac{1}{\alpha}\lg\frac{2\bar{n}}{\sqrt{\pi\alpha}},\quad B_b=\frac{1}{\alpha}\lg\frac{\bar{n}}{2\sqrt{\pi\alpha}F_0},$$

i.e.,

$$\left.\begin{aligned} B_t&=B-\frac{1}{\alpha}\lg\frac{\alpha}{2}\\ B_b&=B-\frac{1}{\alpha}\lg 2\alpha F_0=B_t-\frac{1}{\alpha}\lg 8F_0\end{aligned}\right\}\qquad (30)$$

Hence it follows that

$$B>B_t>B_b. \qquad (31)$$

This means that, with other experimental conditions being equal (for specimens of the same material and identical volume),

$$F_0^*<\tau_0^*<\sigma_0^*. \qquad (32)$$

Thus, technical cohesive strength while bending, σ_0^*, should be greater than technical cohesive strength while twisting, τ_0^*, which in turn should be greater than technical cohesive strength during extension, F_0^*.

As we know, this conclusion agrees well with experimental data.

Substituting in (28) value B_t in the form of (30) and substituting $\alpha = 1/2(\Delta F)^2$, we have

$$\tau_0^*=F_0-\sqrt{A\lg V+B-2\overline{(\Delta F)^2}\lg\frac{1}{4\overline{(\Delta F)^2}}}. \qquad (28')$$

Hence it follows that, just as in the case of bending (cf. (29)), the ratio of technical cohesive strength when twisting to technical cohesive strength during extension, τ_0^*/F_0^*, should be somewhat greater than unity.

Actually, according to Kuntze's data [9], for specimens of gypsum $\tau_0^*/F_0^* = 1.47$; let us recall that according to the data of the same author the ratio of σ_0^*/F_0^* for the same material proved to be equal to 1.83.

Let us note in conclusion that it is possible to assume that the numerical value of ratio τ_0^*/F_0^* (just as with σ_0^*/F_0^*) should be closer to unity, the more uniform the material studied.

Conclusion

The theory presented above does not in any way pretend toward universality. It is only one of many possible variants of a statistical description of the role of different types of defects present in real materials. This variant seems to us, however, one of the most sensible, since as the distribution function choice is made of the Gaussian function, which has recommended itself so well in considering a number of allied questions.

In view of the almost total lack of quantitative experimental data regarding the influence of the scale factor on technical cohesive strength in the nonuniformly stressed state of a material, we are unfortunately for the time being deprived of the opportunity of subjecting our theory to direct experimental verification.

Equations (5) and (6), however, concerning the case of the uniformly stressed state, do not agree too badly with experimental data; qualitative conclusions which it proved possible to make in analyzing the question of technical cohesive strength during bending and extension also agree with experimental data.

This gives us the right to hope that the ideas expounded above can turn out to be not without use in studying the influence of the scale factor on the mechanical characteristics of solids.

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[1]Subsequently, (§ 5) we will provide strict proof of the validity of the assumption that large values of the difference (F_0-F) correspond to samples of large volume and small values of this difference - to samples y small volume.

[2]The tremendous test material which pertains to the study of the increased strength of thin filaments cannot be used to check formulas (5) and (6); we have absolutely different regularities for filament-like samples which also receive a satisfactory explanation within the framework of statistical theory of strength (See [4]).

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